

# STABILITY OF TRAVELLING WAVES IN STOCHASTIC NAGUMO EQUATIONS

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**ABSTRACT.** Stability of travelling waves for the Nagumo equation on the whole line is proven using a new approach via functional inequalities and an implicitly defined phase adaption. The approach can be carried over to obtain pathwise stability of travelling wave solutions in the case of the stochastic Nagumo equation as well. The noise term considered is of multiplicative type with variance proportional to the distance of the solution to the orbit of the travelling wave solutions.

## 1. INTRODUCTION

The purpose of this paper is to introduce a new approach to the study of (local) stability of travelling waves and pulses in excitable media that is in particular well-suited for stochastic perturbations. We are interested in the classical equations modelling the propagation of the action potential travelling along the axon of a neuron. As a starting point in this paper we consider the Nagumo equation on the real line (cf. [10]) perturbed by stochastic forcing terms. We make particular use of the explicit knowledge of the travelling waves in this case. However, our approach will be robust w.r.t. small perturbations in the coefficients.

Since the spectral considerations, employed in the classical stability analysis of nerve axon equations (cf. [3, 5, 6] and the recent monograph [2]) are not easy to carry over to the stochastic case, we look for a pathwise stability analysis in the sense of the classical Lyapunov approach to the stability of dynamical systems. A first novelty of the paper is the introduction of an additional dynamics of gradient type that adapts a given solution of the stochastic Nagumo equation to the correct phase of the travelling wave. This explicitly given phase adaption, which is in addition easy to implement numerically, is the analogue of the phase conditions introduced as algebraic constraints in the classical stability analysis (see in particular [5]). As a second novelty in this paper, we replace the usual spectral considerations, applied to the Schrödinger operator, obtained as linearization of the underlying dynamics along a

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given travelling wave, by functional inequalities of Poincaré type. Our hope is that the latter method will be generalizable also to general systems of reaction diffusion type because it only uses partial information of the travelling wave solutions. Certainly, it is well suited for stochastic perturbations as demonstrated in this paper. An additional advantage is that, in contrast to the usual spectral considerations, our approach allows explicit quantitative estimates, both, in the deterministic and in the stochastic case and sensitivity considerations w.r.t. the coefficients.

The paper is organized as follows: In Section 2 we first present our new approach in the case of the deterministic Nagumo equation, to demonstrate the main arguments in a somewhat easier setting. The analogue to the usual spectral considerations of the Schrödinger operator associated with the linearization along a travelling wave is contained in Theorem 2.3. Theorem 2.6 then contains our main result on the local stability of travelling wave solutions. In Section 3 we consider the Nagumo equation perturbed with multiplicative noise. The variance of the noise term is chosen to be proportional to the distance of the solution w.r.t. orbit generated by the travelling wave of the deterministic Nagumo equation. Hence, combining our stability analysis of Section 2 with a careful analysis of the stochastic perturbation, we obtain in Theorem 3.1 the stochastic analogue of our local stability result in the deterministic case.

In addition to the new approach to the stability analysis via functional inequalities we also would like to mention that the type of stochastic Nagumo equations considered in this paper are also new in comparison with the models of spatially extended neurons subject to noise studied numerically and analytically by Tuckwell and Jost in [12, 13] and also by Lord and Thümmel in [9]. In order to ensure existence and uniqueness of a solution to our stochastic partial differential equation we use the variational approach to stochastic evolution equations as presented in monograph [11] with recent extensions presented in [8]. In particular, we make use of the Itô formula, that can be obtained for the Hilbert space norm of the variational solution. The implied semimartingale decomposition can then be used to apply the one-dimensional (time-dependent) Itô formula to any smooth transformation of the Hilbert norm.

## 2. THE DETERMINISTIC CASE

Consider the Nagumo equation

$$(1) \quad \partial_t v(t, x) = \nu \partial_{xx}^2 v(t, x) + bf(v(t, x)) \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R}$$

on the real line, with  $\nu, b > 0$  and

$$f(v) = v(1-v)(v-a) \quad a \in \left(0, \frac{1}{2}\right).$$

The equation is obtained from the well-known Fitz-Hugh Nagumo system

$$(2) \quad \begin{aligned} \partial_t v(t, x) &= \nu \partial_{xx}^2 v(t, x) + b f(v(t, x)) - w(t, x) + I \\ \partial_t w(t, x) &= \varepsilon (v(t, x) - \gamma w(t, x)) \quad (t, x) \in \mathbb{R}_+ \times \mathbb{R} \end{aligned}$$

by letting  $\varepsilon \downarrow 0$ , i.e., setting the recovery variable  $w$  constant, and further equal to the input current  $I$ . It is well-known that for parameters in the excitable region, the Fitz-Hugh Nagumo system admits a travelling pulse solution modelling signal propagation along the axon of a single neuron. The analogue for the Nagumo equation is a travelling wave front  $v(t, x) = v^{TW}(x + ct)$ , where

$$(3) \quad v^{TW}(x) = \left(1 + \exp\left(-\sqrt{\frac{b}{2\nu}}x\right)\right)^{-1}$$

moving to  $-\infty$  at constant speed  $c = \sqrt{2\nu b}(\frac{1}{2} - a)$  (cf. [1]). We are interested in the local stability of this wave front in the function space  $H = L^2(\mathbb{R})$ .

Before we can state a precise definition of stability, we need to introduce first our concept of a solution that we are working with. To simplify notations in the following we write  $v^{TW}(t) = v^{TW}(\cdot + ct)$ . Next (formally) decompose the function  $v(t, \cdot) = u(t, \cdot) + v^{TW}(t)$  w.r.t. the travelling wave. The resulting equation for  $u$  is then given by

$$(4) \quad \begin{aligned} \partial_t u(t, x) &= \nu \partial_{xx}^2 u(t, x) + b (f(u(t, x) + v^{TW}(t)) - f(v^{TW}(t))) \\ (t, x) &\in \mathbb{R}_+ \times \mathbb{R}. \end{aligned}$$

For the precise definition of the Laplacian  $\partial_{xx}^2$  we need to introduce the Sobolev space  $V = H^{1,2}(\mathbb{R})$  of order 1, equipped with the usual norm  $\|u\|_V^2 := \int (\partial_x u)^2 dx + \|u\|_H^2$ . Clearly,  $V \hookrightarrow H$  densely and continuously. Identifying  $H$  with its dual  $H'$  we obtain the embeddings  $V \hookrightarrow H \equiv H' \hookrightarrow V'$ . Recall that w.r.t. this embedding the dualization  ${}_{V'}\langle f, u \rangle_V$  between  $f \in V'$  and  $u \in V$  reduces to  ${}_{V'}\langle f, u \rangle_V = \langle f, v \rangle_H = \int f u dx$ , i.e. the scalar product in  $H$  in the case where  $f \in H$ . The Laplacian  $\partial_{xx}^2$  then induces a linear continuous mapping  $\Delta : V \rightarrow V'$ , since  ${}_{V'}\langle \Delta u, v \rangle_V = - \int \partial_x u \partial_x v dx \leq \|u\|_V \|v\|_V$ .

The nonlinear term

$$(5) \quad G(t, u) := f(u(t, x) + v^{TW}(t)) - f(v^{TW}(t))$$

in equation (4) can be realized as a continuous mapping

$$G(\cdot, \cdot) : [0, \infty] \times V \rightarrow V'$$

that is Lipschitz w.r.t. the second variable on bounded subsets of  $V$  with Lipschitz constant independent of  $t$ . Indeed, due to the elementary estimate  $\|u\|_\infty \leq \|u\|_V$ , the Taylor representation

$$\begin{aligned} G(t, u) &= f(u + v^{TW}(t)) - f(v^{TW}(t)) \\ &= f'(v^{TW}(t))u + \frac{1}{2}f^{(2)}(v^{TW}(t))u^2 + \frac{1}{6}f^{(3)}(v^{TW}(t))u^3 \end{aligned}$$

and uniform bounds on  $\|f^{(k)}(v^{TW}(t))\|_\infty$ ,  $k = 1, 2, 3$ , we have for  $w \in V$  that

$$\begin{aligned} \langle G(t, u), w \rangle &\leq \int |f(u + v^{TW}(t)) - f(v^{TW}(t))| |w| dx \\ &\leq c_1 \|u\|_V (1 + \|u\|_H^2) \|w\|_V, \end{aligned}$$

hence

$$(6) \quad \|G(t, u)\|_{V'} \leq c_1 \|u\|_V (1 + \|u\|_H^2)$$

and

$$\begin{aligned} \langle G(t, u_1) - G(t, u_2), w \rangle &\leq \int |f(u_1 + v^{TW}(t)) - f(u_2 + v^{TW}(t))| |w| dx \\ &\leq c_2 (1 + \|u_1\|_V^2 + \|u_2\|_V^2) \|u_1 - u_2\|_H \|w\|_V \end{aligned}$$

so that

$$(7) \quad \|G(t, u_1) - G(t, u_2)\|_{V'} \leq c_2 (1 + \|u_1\|_V^2 + \|u_2\|_V^2) \|u_1 - u_2\|_H$$

for finite constants  $c_1$  and  $c_2$  depending on  $f_{|[0,1]}$  only.

Note also that the sum  $\nu \Delta u + bG(t, u)$  satisfies the (global) monotonicity condition

$$(8) \quad \langle \nu \Delta u_1 + bG(t, u_1) - \nu \Delta u_2 - bG(t, u_2), u_1 - u_2 \rangle \leq b\eta \|u_1 - u_2\|_H^2$$

where  $\eta = \sup_{\xi \in \mathbb{R}} f'(\xi) = \frac{1-a+a^2}{3}$ , since  $(f(s) - f(t))(s - t) \leq \eta(s - t)^2$  for all  $s, t \in \mathbb{R}$ , and the coercivity condition

$$(9) \quad \langle \nu \Delta u + bG(t, u), u \rangle \leq -\nu \|u\|_V^2 + (b\eta + \nu) \|u\|_H^2$$

since  $f(s)s = (f(s) - f(0))(s - 0) \leq \eta s^2$  for all  $s \in \mathbb{R}$ .

It is now standard (see, e.g. Theorem 1.1 in [8]) to deduce for all  $u_0 \in H$  and all finite times  $T$  existence and uniqueness of a variational solution  $u \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$  satisfying the integral equation

$$(10) \quad u(t) = u_0 + \int_0^t \nu \Delta u(s, \cdot) + bf(u(s) + v^{TW}(s)) - bf(v^{TW}(s)) ds$$

associated with (4). Clearly, we may consider this solution  $u$  as a solution on the whole time axes  $t \geq 0$ .

The integral  $\int_0^t \nu \Delta u(s, \cdot) + bf(u(s) + v^{TW}(s)) - bf(v^{TW}(s)) ds$  appearing in the integral equation (10) is well-defined as a Bochner integral in  $L^2([0, T]; V')$ , since due to (6)

$$\begin{aligned} & \int_0^t \|\Delta u(s)\|_{V'}^2 + \|f(u(s) + v^{TW}(s)) - f(v^{TW}(s))\|_{V'}^2 ds \\ & \leq c \int_0^t \|u(s)\|_V^2 ds \left( 1 + \sup_{t \in [0, T]} \|u(t)\|_H^4 \right) < \infty \end{aligned}$$

for all  $t \geq 0$ . In particular, the mapping  $t \mapsto u(t)$ ,  $[0, \infty) \rightarrow V'$ , is differentiable with differential

$$\frac{du(t)}{dt} = \nu \Delta u(t) + bf(u(t) + v^{TW}(t)) - f(v^{TW}(t)) \in V'$$

and therefore also locally Lipschitz.

*Definition 2.1.* The travelling wave solution  $v^{TW}$  is called locally stable in  $H$  if there exists  $\delta > 0$  such that for initial condition  $v_0$  with  $v_0 - v^{TW} \in H$  and  $\|v_0 - v^{TW}\|_H \leq \delta$  the (unique variational) solution  $u(t, x)$  to (4) satisfies

$$\lim_{t \rightarrow \infty} \|v(t, \cdot) - v^{TW}(t + t_0)\|_H = 0$$

for some (phase)  $t_0 \in \mathbb{R}$ .

The stability of travelling wave fronts for the Nagumo equation has been studied in many papers as a prototype example for metastability. The mathematical analysis of stability properties of  $v^{TW}$  faces two major difficulties. The first one is the obvious fact that the reaction term  $f(u)$  in the equation (1) is not strictly dissipative in the sense that

$$\langle \nu \partial_{xx}^2 v_1 + bf(v_1) - \nu \partial_{xx}^2 v_2 - bf(v_2), v_1 - v_2 \rangle \leq -\kappa_* \|v_1 - v_2\|_H^2$$

for some  $\kappa_* > 0$ , or equivalently, the associated potential  $F(v) = \int_{v_0}^v f(t) dt$  is not uniformly strictly convex, but a double-well potential. This remains true if we fix  $v_2$  to be equal to the travelling wave  $v^{TW}$  or any of its spatial translates  $v^{TW}(\cdot - y)$ . A first naive calculation, exploiting the coercivity condition (9), only yields the following a priori estimate.

**Lemma 2.2.** *Let  $u \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$  be the unique solution of (4). Then*

$$\|u(t)\|_H^2 \leq e^{2b\eta t} \|u_0\|_H^2 \quad \forall t \in [0, T].$$

*Proof.* The coercivity condition (9) implies that

$$\begin{aligned} \frac{d}{dt} \|u(t)\|_H^2 &= 2 \langle \nu \Delta u(t) + bG(t, u(t)), u(t) \rangle \\ &\leq 2b\eta \|u(t)\|_H^2 \quad t \in [0, T]. \end{aligned}$$

Integrating up the last inequality w.r.t.  $t$  yields the desired inequality.  $\square$

However, restricting  $u$  to the orthogonal component of the derivative  $\partial_x v^{TW}$  of the travelling wave solution, i.e.  $\int u \partial_x v^{TW} dx = 0$ , we have, under suitable assumptions on  $a$  the local dissipativity

$$\langle \nu \partial_{xx}^2 u, u \rangle + b \langle f'(v^{TW})u, u \rangle \leq -\kappa_* \|u\|_V^2 + C_1 \|u\|_V \|u\|_H^2$$

for some  $\kappa_* > 0$  and some finite constant  $C_1$  which is implied by the following Theorem:

**Theorem 2.3.** *Assume that  $a > \frac{1}{2} \left(1 - \sqrt{\frac{3}{10}}\right)$ , hence  $\kappa_*(a) = \frac{10}{3}(1 - 2a)^2 < 1$ . Let*

$$\kappa_* = \frac{3b}{6b + 40} \frac{1 - \kappa_*(a)}{1 + b(\eta + 1)} \quad \text{and} \quad C_* = 3b \sqrt{\frac{b}{2\nu}}.$$

Then

$$\langle \nu \partial_{xx}^2 u + b \langle f'(v^{TW})u, u \rangle \leq -\kappa_* \|u\|_V^2 + C_* \langle u, \partial_x v^{TW} \rangle^2$$

for all  $u \in V$ .

The proof of the Theorem is postponed to Section 4.

The second difficulty in the mathematical analysis of the stability properties of  $v^{TW}$  is to identify the correct phase-shift of  $v^{TW}(t + t_0)$  to which to compare the given solution  $v$  of (1). To this end we introduce an auxiliary ordinary differential equation of gradient descent type associated with the minimization of the distance between  $v$  and the set  $\mathcal{N} = \{v^{TW}(\cdot + C) \mid C \in \mathbb{R}\}$  of all phase-shifted travelling waves. More precisely, given a solution  $v$  to the Nagumo equation (1) with initial condition  $v_0$  satisfying  $v_0 - v^{TW} \in H^{1,2}(\mathbb{R})$ , and given any relaxation rate  $m > 0$  (that will be specified later) we consider the ordinary differential equation

$$\begin{aligned} (11) \quad \dot{C}(t) &= -m \langle \partial_x v^{TW}(\cdot + C(t) + ct), v^{TW}(\cdot + C(t) + ct) - v(t, \cdot) \rangle_H \\ C(0) &= 0. \end{aligned}$$

The next Proposition states that the ordinary differential equation is well-posed.

**Proposition 2.4.** *Let  $v = u + v^{TW}(t)$  be a solution to (1) with  $u \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$ . Then*

$$B(t, C) = \langle \partial_x v^{TW}(\cdot + C + ct), v^{TW}(\cdot + C + ct) - v(t, \cdot) \rangle_H$$

*is continuous in  $(t, C) \in [0, T] \times \mathbb{R}$ , and Lipschitz continuous w.r.t.  $C$  with Lipschitz constant independent of  $t$ .*

*Proof.* Using the representation

$$\begin{aligned} B(t, C) &= \langle \partial_x v^{TW}(\cdot + C + ct), v^{TW}(\cdot + C + ct) - v^{TW}(t) \rangle_H \\ &\quad + \langle \partial_x v^{TW}(\cdot + C + ct), v^{TW}(\cdot + ct) - v(t, \cdot) \rangle_H \end{aligned}$$

the continuity of  $B$  follows from the continuity of  $(t, C) \mapsto \partial_x v^{TW}(\cdot + C + ct)$  as a mapping with values in  $V$ , the continuity of  $(t, C) \mapsto v^{TW}(\cdot + C + ct) - v^{TW}(t)$  as a mapping with values in  $V'$  and the (Lipschitz) continuity of  $t \mapsto v(t, \cdot) - v^{TW}(t)$  as a mapping with values in  $V'$ .

For the proof of the Lipschitz property w.r.t.  $C$  note that

$$\begin{aligned} B(t, C_1) - B(t, C_2) &= \langle \partial_x v^{TW}(\cdot + C_1 + ct) - \partial_x v^{TW}(\cdot + C_2 + ct), v^{TW}(\cdot + ct) - v(t, \cdot) \rangle_H \\ &\quad + \langle \partial_x v^{TW}(\cdot + C_1 + ct), v^{TW}(\cdot + C_1 + ct) - v^{TW}(\cdot + ct) \rangle_H \\ &\quad - \langle \partial_x v^{TW}(\cdot + C_2 + ct), v^{TW}(\cdot + C_2 + ct) - v^{TW}(\cdot + ct) \rangle_H \\ &= I + II + III, \text{ say.} \end{aligned}$$

We next assume w.l.o.g.  $C_1 \leq C_2$ . Then

$$\begin{aligned} (12) \quad &|v^{TW}(x + C_1 + ct) - v^{TW}(x + C_2 + ct)| \\ &= \sqrt{\frac{b}{2\nu}} \int_{C_1}^{C_2} \frac{\exp(-\sqrt{\frac{b}{2\nu}}(x + \xi + ct))}{(1 + \exp(-\sqrt{\frac{b}{2\nu}}(x + \xi + ct)))^2} d\xi \\ &= \sqrt{\frac{b}{2\nu}} \int_{C_1}^{C_2} \exp(-\sqrt{\frac{b}{2\nu}}(x + \xi + ct)) v^{TW}(x + \xi + ct)^2 d\xi, \end{aligned}$$

so that we can further estimate

$$\begin{aligned} (13) \quad &|II + III| = |\langle \partial_x v^{TW}(\cdot + ct), v^{TW}(\cdot + ct) - v^{TW}(\cdot - C_1 + ct) \rangle_H \\ &\quad - \langle \partial_x v^{TW}(\cdot + ct), v^{TW}(\cdot + ct) - v^{TW}(\cdot - C_2 + ct) \rangle_H| \\ &= |\langle \partial_x v^{TW}(\cdot + ct), v^{TW}(\cdot - C_1 + ct) - v^{TW}(\cdot - C_2 + ct) \rangle_H| \\ &\leq \|\partial_x v^{TW}(\cdot + ct)\|_H \\ &\quad \cdot \sqrt{\frac{b}{2\nu}} \int_{C_1}^{C_2} \|\exp(-\sqrt{\frac{b}{2\nu}}(\cdot + \xi + ct)) v^{TW}(\cdot + \xi + ct)^2\|_H d\xi \\ &\leq \text{const} \cdot |C_1 - C_2|. \end{aligned}$$

Similarly, using

$$|\partial_{xx}^2 v^{TW}(x)| \leq 3\sqrt{\frac{b}{2\nu}} |\partial_x v^{TW}(x)|$$

$$\begin{aligned}
(14) \quad |I| &\leq \int_{C_1}^{C_2} \int_{\mathbb{R}} |\partial_{xx}^2 v^{TW}(x + \xi + ct) u(t, x)| \, dx \, d\xi \\
&\leq 3 \sqrt{\frac{b}{2\nu}} \int_{C_1}^{C_2} \|\partial_x v^{TW}(\cdot + \xi + ct)\|_H \|u(t)\|_H \, d\xi \\
&\leq 3 \sqrt{\frac{b}{2\nu}} |C_1 - C_2| \sup_{t \in [0, T]} \|u(t)\|_H.
\end{aligned}$$

Inserting (13) and (14) into (12) yields the desired assertion.  $\square$

According to the last Proposition the function  $C$  defined by (11) is well-defined. As already indicated,  $C$  will adapt to the correct phase of the  $v$  if we choose  $m \geq C_*$  (cf. 2.3) and our aim is to prove in the following that the difference

$$(15) \quad \tilde{u}(t) := u(t) + v^{TW}(t) - v^{TW}(\cdot + C(t) + ct) = v(t) - v^{TW}(\cdot + C(t) + ct)$$

converges to zero as  $t \rightarrow \infty$  if the initial condition  $u_0 = v_0 - v^{TW}$  is sufficiently small in the  $H$ -norm. In the next Proposition we first identify the resulting evolution equation for  $\tilde{u}$ .

**Proposition 2.5.** *Let  $u = v - v^{TW}(t) \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$  be a solution of (4) and let  $\tilde{u}$  be defined by (15). Then  $\tilde{u} \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$  again and  $\tilde{u}$  satisfies the evolution equation*

$$\begin{aligned}
(16) \quad \frac{d\tilde{u}}{dt}(t) &= \nu \Delta \tilde{u}(t) + b \tilde{G}(t, \tilde{u}(t)) - \dot{C}(t) \partial_x v^{TW}(\cdot + C(t) + ct) \\
&= \nu \Delta \tilde{u}(t) + b \tilde{G}(t, \tilde{u}(t)) \\
&\quad - m \langle \partial_x v^{TW}(\cdot + C(t) + ct), \tilde{u}(t) \rangle \partial_x v^{TW}(\cdot + C(t) + ct)
\end{aligned}$$

with

$$\tilde{G}(t, u) = f(u + v^{TW}(\cdot + C(t) + ct)) - f(v^{TW}(\cdot + C(t) + ct)).$$

In particular,

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_H^2 &= -\nu \|\partial_x u(t)\|_H^2 + b \langle \tilde{G}(t, \tilde{u}(t)), \tilde{u}(t) \rangle \\
&\quad - m \langle \partial_x v^{TW}(\cdot + C(t) + ct), \tilde{u}(t) \rangle^2.
\end{aligned}$$

The proof of the Proposition is an immediate consequence of the properties of  $v^{TW}$  and the equations (4) and (11).

As usual we will now consider the linearization of the mapping  $\tilde{G}(t, u)$  around zero. To simplify notations, let  $\tilde{v}^{TW}(t) := v^{TW}(\cdot + C(t) + ct)$ . Then we can write

$$(17) \quad \tilde{G}(t, u) = f'(\tilde{v}^{TW}(t))u + \tilde{R}(t, u)$$

where

$$\begin{aligned}\tilde{R}(t, u) &= f(u + \tilde{v}^{TW}(t)) - f(\tilde{v}^{TW}) - f'(\tilde{v}^{TW}(t))u \\ &= \frac{1}{2}f^{(2)}(\tilde{v}^{TW}(t))u^2 + \frac{1}{6}f^{(3)}(\tilde{v}^{TW}(t))u^3\end{aligned}$$

satisfies the estimates

$$(18) \quad \langle \tilde{R}(t, u), u \rangle \leq (4 + a)\|u\|_H^2\|u\|_V \leq (4 + a)\|u\|_H\|u\|_V^2$$

and

$$(19) \quad \|\tilde{R}(t, u)\|_{V'} \leq (4 + a)\|u\|_H^2(1 + \|u\|_V)$$

Similar to the classical stability analysis of the Nagumo equation we now use the information on the spectrum of the Schrödinger operator  $\nu\Delta u + bf'(\tilde{v}^{TW})u$  contained in Theorem 2.3 with the above localization to obtain the first local stability result.

### 2.1. Main result.

**Theorem 2.6.** *Assume that  $a > \frac{1}{2} \left(1 - \sqrt{\frac{3}{10}}\right)$ . Recall the definition of  $\kappa_*$  and  $C_*$  in Theorem 2.3. If the initial condition  $v_0 = u_0 + v^{TW}$  is close to  $v^{TW}$  in the sense that*

$$\|u_0\|_H < \delta \frac{\kappa_*}{b(4 + a)}$$

for some  $\delta < 1$  and  $v(t) = u(t) + v^{TW}(t)$ , where  $u(t)$  is the unique solution of (4), then

$$\|v(t) - v^{TW}(\cdot + C(t) + ct)\|_H \leq e^{-(1-\delta)\kappa_*t} \|v_0 - v^{TW}\|_H.$$

Here,  $C(t)$  is the solution of (11) with  $m \geq C_*$ .

*Proof.* Let  $\tilde{u}(t) := v(t) - \tilde{v}^{TW}(t)$  be as in (15). Then Proposition 2.5 and (18) imply that

$$\begin{aligned}(20) \quad \frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_H^2 &= \langle \nu\Delta \tilde{u}(t) + bf'(\tilde{v}^{TW}(t))\tilde{u}(t), \tilde{u}(t) \rangle + b\langle \tilde{R}(t, \tilde{u}(t)), \tilde{u}(t) \rangle \\ &\quad - m \left( \langle \partial_x \tilde{v}^{TW}(t), \tilde{u}(t) \rangle \right)^2 \\ &\leq \langle \nu\Delta \tilde{u}(t) + bf'(\tilde{v}^{TW}(t))\tilde{u}(t), \tilde{u}(t) \rangle \\ &\quad + b(4 + a)\|\tilde{u}(t)\|_H\|\tilde{u}(t)\|_V^2 - m \left( \langle \partial_x \tilde{v}^{TW}(t), \tilde{u}(t) \rangle \right)^2.\end{aligned}$$

Using translation invariance of  $\nu\Delta$  and  $\int (\partial_x u)^2 dx$ , Theorem 2.3 yields the estimate

$$\begin{aligned}(21) \quad &\langle \nu\Delta \tilde{u}(t) + bf'(\tilde{v}^{TW}(t))\tilde{u}(t), \tilde{u}(t) \rangle \\ &\leq -\kappa_*\|\tilde{u}(t)\|_V^2 + C_* \left( \int \tilde{u}(t) \partial_x \tilde{v}^{TW} dx \right)^2.\end{aligned}$$

Inserting (21) into (20) yields that

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_H^2 \leq -\kappa_* \|\tilde{u}(t)\|_V^2 + b(4+a) \|\tilde{u}(t)\|_H \|\tilde{u}(t)\|_V^2.$$

In the next step we define the stopping time

$$T := \inf \left\{ t \geq 0 \mid \|\tilde{u}(t)\|_H \geq \delta \frac{\kappa_*}{b(4+a)} \right\}$$

with the usual convention  $\inf \emptyset = \infty$ . Continuity of  $t \mapsto \|\tilde{u}(t)\|_H$  implies that  $T > 0$  since  $\|u_0\|_H < \delta \frac{\kappa_*}{b(4+a)}$ . For  $t < T$  note that

$$\frac{1}{2} \frac{d}{dt} \|\tilde{u}(t)\|_H^2 \leq -(1-\delta)\kappa_* \|\tilde{u}(t)\|_V^2 \leq -(1-\delta)\kappa_* \|\tilde{u}(t)\|_H^2$$

which implies that

$$\|\tilde{u}(t)\|_H^2 \leq e^{-2(1-\delta)\kappa_* t} \|u_0\|_H^2$$

for  $t < T$ . Suppose now that  $T < \infty$ . Then continuity of  $t \mapsto \|\tilde{u}(t)\|_H$  implies on the one hand that  $\|\tilde{u}(T)\|_H = \delta \frac{\kappa_*}{b(4+a)}$  and on the other hand, using the last inequality,

$$\|\tilde{u}(T)\|_H = \lim_{t \uparrow T} \|\tilde{u}(t)\|_H \leq e^{-(1-\delta)\kappa_* T} \|u_0\|_H < \delta \frac{\kappa_*}{b(4+a)}$$

which is a contradiction. Consequently,  $T = \infty$  and thus

$$\|\tilde{u}(t)\|_H \leq e^{-(1-\delta)\kappa_* t} \|u_0\|_H \quad \forall t \geq 0$$

which implies the assertion.  $\square$

### 3. STOCHASTIC STABILITY

We now turn to the stochastic Nagumo equation

$$(22) \quad dv(t) = [\nu \partial_{xx}^2 v(t) + bf(v(t))] dt + \Sigma(v(t)) dW(t)$$

with a multiplicative noise term.  $W = (W(t))_{t \geq 0}$  is a cylindrical Wiener process with values in some separable real Hilbert space  $U$ , defined on some underlying filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \geq 0}, P)$ . The theory of Wiener processes on Hilbert spaces and associated stochastic evolution equations can be found in the monograph [11]. The dispersion coefficient  $\Sigma$ , describing the standard deviation of the noise term, formally is a function of the  $L^2$ -distance  $\inf_{C \in \mathbb{R}} \|v - v^{TW}(\cdot + C)\|_H$  of  $v$  to the set  $\mathcal{N}$  of travelling waves, i.e.,  $\Sigma(v) = \Sigma(v - v^{TW}(\cdot + C))$  for all  $C \in \mathbb{R}$ .

As in the deterministic case we will give the equation a rigorous meaning by decomposing  $v(t) = u(t) + v^{TW}(t)$  w.r.t. the travelling wave. The stochastic evolution equation for  $u$  is then given by

$$(23) \quad du(t) = [\nu \Delta u(t) + bG(t, u(t))] dt + \Sigma(u(t)) dW(t)$$

where the nonlinear term  $G$  is as in (5) and we will now make the following assumptions on  $\Sigma$ .  $\Sigma : H \mapsto L_2(U, H)$  satisfies

$$(24) \quad \Sigma(u) = \Sigma(u + v^{TW} - v^{TW}(\cdot + C)) \text{ for any } C \in \mathbb{R}, \Sigma(0) = 0$$

and  $\Sigma$  is Lipschitz continuous

$$(25) \quad \|\Sigma(v_1) - \Sigma(v_2)\|_{L_2(U, H)}^2 \leq \sigma^2 \|v_1 - v_2\|_H^2.$$

for some constant  $\sigma^2$ . Here,  $L_2(U, H)$  denotes the space of all Hilbert-Schmidt operators  $L : U \mapsto H$  with the Hilbert-Schmidt norm  $\|L\|_{L_2(U, H)}^2 = \sum_k \|Le_k\|_H^2$  for one (hence any) complete orthonormal system  $(e_k)_k$  of  $U$ .

We consider the equation (23) w.r.t. the same triple  $V \hookrightarrow H \equiv H' \hookrightarrow V'$  as in Section 2. Due to the properties (6), (7), (8) and (9), we can now deduce from Theorem 1.1. in [8] for all finite  $T$  and all (deterministic) initial conditions  $u_0 \in H$  the existence and uniqueness of a solution  $(u(t))_{t \in [0, T]}$  of (23) satisfying the moment estimate

$$E \left( \sup_{t \in [0, T]} \|u(t)\|_H^2 + \int_0^T \|u(t)\|_V^2 dt \right) < \infty$$

which implies in particular that  $u \in L^\infty([0, T]; H) \cap L^2([0, T]; V)$  P-a.s. As a consequence we can apply Proposition 2.4 to a typical trajectory  $u(\cdot)(\omega)$  to obtain a unique solution  $C(\cdot)(\omega)$  of equation (11). It is also clear that the resulting stochastic process  $(C(t))_{t \geq 0}$  is  $(\mathcal{F}_t)_{t \geq 0}$ -adapted, since  $(u(t))_{t \geq 0}$  is. We will assume as in the deterministic case that the relaxation rate  $m$  is sufficiently large, i.e.,  $m > C_*$ .

Similar to the deterministic case we now define the stochastic process

$$\tilde{u}(t) = u(t) + v^{TW}(t) - v^{TW}(\cdot + C(t) + ct) = v(t) - \tilde{v}^{TW}(t)$$

which is  $(\mathcal{F}_t)_{t \geq 0}$  adapted too and satisfies the stochastic evolution equation

$$d\tilde{u}(t) = \left[ \nu \Delta \tilde{u}(t) + b\tilde{G}(t, \tilde{u}(t)) - \dot{C}(t) \partial_x \tilde{v}^{TW}(t) \right] dt + \Sigma(\tilde{u}(t)) dW(t)$$

where

$$\tilde{G}(t, u) = f(u + \tilde{v}^{TW}(t)) - f(\tilde{v}^{TW}(t))$$

and the moment estimates

$$E \left( \sup_{t \in [0, T]} \|\tilde{u}(t)\|_H^2 + \int_0^T \|\tilde{u}(t)\|_V^2 dt \right) < \infty.$$

Due to [11], Theorem 4.2.5, we have the Ito-formula

$$\begin{aligned} \|\tilde{u}\|_H^2(t) &= \|\tilde{u}(0)\|_H^2 + \int_0^t 2 \langle \nu \Delta \tilde{u}(s) + \tilde{G}(s, \tilde{u}(s)) \\ &\quad - \dot{C}(s) \partial_x \tilde{v}^{TW}(s), \tilde{u}(s) \rangle + \|\Sigma(\tilde{u}(s))\|_{L_2(U, H)}^2 ds + \tilde{M}_t \end{aligned}$$

with

$$\tilde{M}_t = 2 \int_0^t \langle \tilde{u}(s), \Sigma(\tilde{u}(s)) dW(s) \rangle.$$

It follows from the above representation that  $\|\tilde{u}(t)\|_H^2$  is a (scalar-valued) continuous local semimartingale, in particular we have also the (one-dimensional) time-dependent Ito-formula

$$\begin{aligned} \varphi(t, \|\tilde{u}(t)\|_H^2) &= \int_0^t \partial_t \varphi(s, \|\tilde{u}(s)\|_H^2) + 2\partial_x \varphi(s, \|\tilde{u}(s)\|_H^2) \langle \nu \Delta \tilde{u}(s) \\ &\quad + b\tilde{G}(s, \tilde{u}(s)) - \dot{C}(s) \partial_x \tilde{v}^{TW}(s), \tilde{u}(s) \rangle \\ (26) \quad &\quad + \partial_x \varphi(s, \|\tilde{u}(s)\|_H^2) \|\Sigma(\tilde{u}(s))\|_{L_2(U, H)}^2 \\ &\quad + \partial_{xx}^2 \varphi(s, \|\tilde{u}(s)\|^2) 2\|\Sigma^*(\tilde{u}(s))\tilde{u}(s)\|_U^2 ds \\ &\quad + \int_0^t \partial_x \varphi(s, \|\tilde{u}(s)\|_H^2) d\tilde{M}_s \end{aligned}$$

for any  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}_+)$ . Here,  $\Sigma^*(u)$  denotes the Hilbert-Schmidt operator from  $H$  to  $U$  adjoint to  $\Sigma(u)$ .

**Theorem 3.1.** *Assume that  $a > \frac{1}{2} \left(1 - \sqrt{\frac{3}{10}}\right)$ . Recall the definition of  $\kappa_*$  and  $C_*$  in Theorem 2.3 and assume that  $\sigma^2 \leq \frac{\kappa_*}{2}$ . Let  $v_0 = u_0 + v^{TW}$ . Let  $v(t) = u(t) + v^{TW}(t)$ , where  $u(t)$  is the unique solution of the stochastic evolution equation (23) and  $\tilde{u}(t) = u(t) + v^{TW}(t) - \tilde{v}^{TW}(t)$ . Let*

$$(27) \quad T := \inf\{t \geq 0 \mid \|\tilde{u}(t)\|_H > c_*\}, \quad c_* = \frac{\kappa_*}{2b(4+a)},$$

with the usual convention  $\inf \emptyset = \infty$ . Then

$$P(T < \infty) \leq \frac{1}{c_*^2} \|\tilde{u}(0)\|_H^2$$

and

$$\lim_{t \rightarrow \infty} \|\tilde{u}(t)\|_H = 0$$

on the set  $\{T = +\infty\}$

*Proof.* Similar to the proof of Theorem 2.6 we have the following inequality

$$\begin{aligned} &\langle \nu \Delta \tilde{u}(t) + b\tilde{G}(t, \tilde{u}(t)) - \dot{C}(t) \partial_x \tilde{v}^{TW}(t), \tilde{u}(t) \rangle \\ &\leq -\kappa_* \|\tilde{u}(t)\|_V^2 + b(4+a) \|\tilde{u}(t)\|_H \|\tilde{u}(t)\|_V^2. \end{aligned}$$

In particular,

$$\langle \nu \Delta \tilde{u}(t) + b\tilde{G}(t, \tilde{u}(t)) - \dot{C}(t) \partial_x \tilde{v}^{TW}(t), \tilde{u}(t) \rangle \leq -\frac{\kappa_*}{2} \|\tilde{u}(t)\|_V^2$$

for  $t \leq T$ , where  $T$  is as in (27). Since also  $\|\Sigma(\tilde{u}(t))\|_{L_2(U,H)}^2 \leq \sigma^2 \|\tilde{u}(t)\|_H^2$  it follows from the choice of  $\sigma$  that

$$(28) \quad 2\langle \nu \Delta \tilde{u}(t) + b\tilde{G}(t, \tilde{u}(t)) - \dot{C}(t) \partial_x \tilde{v}^{TW}(t), \tilde{u}(t) \rangle + \|\Sigma(\tilde{u}(t))\|_{L_2(U,H)}^2 \leq -\frac{\kappa_*}{2} \|\tilde{u}(t)\|_V^2$$

Applying Ito's formula (26) to  $\log(\varepsilon + x)$ , (28) implies for  $t < T$  that

$$\begin{aligned} \log(\varepsilon + \|\tilde{u}(t)\|_H^2) &\leq \log(\varepsilon + \|\tilde{u}(0)\|_H^2) - \frac{\kappa_*}{2} \int_0^t \frac{\|\tilde{u}(s)\|_V^2}{\varepsilon + \|\tilde{u}(s)\|_H^2} ds \\ &\quad + M_t^{(\varepsilon)} - \frac{1}{2} \langle M^{(\varepsilon)} \rangle_t \end{aligned}$$

where

$$M_t^{(\varepsilon)} = 2 \int_0^t \frac{\langle \tilde{u}(s), \Sigma(\tilde{u}(s)) dW(s) \rangle}{\varepsilon + \|\tilde{u}(s)\|_H^2}.$$

It follows that

$$\varepsilon + \|\tilde{u}(t)\|_H^2 \leq (\varepsilon + \|\tilde{u}(0)\|_H^2) \exp \left( -\frac{\kappa_*}{2} \int_0^t \frac{\|\tilde{u}(s)\|_H^2}{\varepsilon + \|\tilde{u}(s)\|_H^2} ds + M_t^{(\varepsilon)} - \frac{1}{2} \langle M^{(\varepsilon)} \rangle_t \right).$$

We can now take the limit  $\varepsilon \rightarrow 0$  to obtain the pathwise inequality

$$\|\tilde{u}(t)\|_H^2 \leq \|\tilde{u}(0)\|_H^2 \exp \left( -\frac{\kappa_*}{2} t + M_t - \frac{1}{2} \langle M \rangle_t \right)$$

where

$$M_t = 2 \int_0^t \frac{\langle \tilde{u}(s), \Sigma(\tilde{u}(s)) dW(s) \rangle}{\|\tilde{u}(s)\|_H^2}.$$

Since  $T$  is a stopping time and  $\exp(M_t - \frac{1}{2} \langle M \rangle_t)$  a nonnegative continuous local martingale, hence a supermartingale, the optional sampling theorem now implies that

$$\begin{aligned} c_*^2 E \left( e^{\frac{\kappa_*}{2} T} 1_{\{T < \infty\}} \right) &\leq \lim_{t \uparrow \infty} E \left( e^{\frac{\kappa_*}{2} T \wedge t} \|\tilde{u}(T \wedge t)\|_H^2 \right) \\ &\leq \lim_{t \uparrow \infty} E \left( \exp(M_{T \wedge t} - \frac{1}{2} \langle M \rangle_{T \wedge t}) \right) \|\tilde{u}(0)\|_H^2 \\ &\leq \|\tilde{u}(0)\|_H^2. \end{aligned}$$

In particular

$$P(T < \infty) \leq \frac{1}{c_*^2} \|\tilde{u}(0)\|_H^2.$$

On the other hand note that on the set  $\{T = \infty\}$  clearly

$$\|\tilde{u}(t)\|_H^2 \leq \|\tilde{u}(0)\|_H^2 \exp \left( -\frac{\kappa_*}{2} t + M_t - \frac{1}{2} \langle M \rangle_t \right) \rightarrow 0$$

because  $(M_t)_{t \geq 0}$  is a time-changed Brownian motion (see [7], Theorem 3.4.6 and Problem 3.4.7), i.e., there exists a Brownian motion  $(B_t)_{t \geq 0}$  on a possibly extended probability space such that  $M_t = B_{\langle M \rangle_t}$ , and thus

$\limsup_{t \rightarrow \infty} M_t - \frac{1}{2} \langle M \rangle_t = \limsup_{t \rightarrow \infty} \langle M \rangle_t \left( \frac{B_{\langle M \rangle_t}}{\langle M \rangle_t} \right) - \frac{1}{2} \langle M \rangle_t < +\infty$ ,  
which implies the assertion.  $\square$

#### 4. PROOF OF THEOREM 2.3

The proof of Theorem 2.3 requires a number of preliminary results. To simplify notations in the following we simply write  $v$  instead of  $v^{TW}$  in the whole section. Let  $w(x) = v(1-v)(x) = \frac{e^{-kx}}{(1+e^{-kx})^2}$  with  $k = \sqrt{\frac{b}{2\nu}}$ .

**Proposition 4.1.** *Let  $u \in C_c^1(\mathbb{R})$  and write  $u = hw$ . Then*

$$\begin{aligned} \langle \nu \Delta u + bf'(v)u, u \rangle &\leq -\frac{\nu}{2} \int (\partial_x h)^2 w^2 dx + \frac{c^2}{2\nu} \int h^2 w^2 dx \\ &\leq -(1 - \kappa_*(a)) \frac{\nu}{2} \int (\partial_x h)^2 w^2 dx + \frac{b}{4} 6k \langle h, w^2 \rangle^2. \end{aligned}$$

Here

$$\kappa_*(a) = \frac{10}{3}(1 - 2a)^2.$$

Moreover, if  $a > \max\{\frac{1}{2}(1 - \sqrt{\frac{3}{10}}), 0\} \sim 0.226$ , then  $\kappa_*(a) < 1$  and

$$\begin{aligned} \langle \nu \Delta u + bf'(v)u, u \rangle &\leq -\frac{3b}{6b + 40}(1 - \kappa_*(a)) \left( \nu \int (\partial_x h)^2 w^2 dx + \|u\|_H^2 \right) \\ &\quad + 3b \sqrt{\frac{b}{2\nu}} \langle h, w^2 \rangle^2. \end{aligned}$$

*Proof.* First note that

$$\nu \Delta u + bf'(v)u = \left( \nu \Delta h + 2\nu \frac{\nabla w}{w} \nabla h + c \frac{\nabla w}{w} h \right) w$$

because

$$\nu \Delta w + f'(v)w = c \nabla w.$$

Integrating against  $u dx = hw dx$  yields

$$\begin{aligned} \langle \nu \Delta u + bf'(v)u, u \rangle &= \int \left( \nu \Delta h + 2\nu \frac{\nabla w}{w} \nabla h \right) hw^2 dx + c \int h^2 \nabla w w dx \\ &= -\nu \int (\partial_x h)^2 w^2 dx - c \int h \nabla h w^2 dx \\ &\leq -\frac{\nu}{2} \int (\partial_x h)^2 w^2 dx + \frac{c^2}{2\nu} \int h^2 w^2 dx. \end{aligned}$$

We will prove in Lemma 4.2 below that

$$\int h^2 w^2 dx \leq \kappa \int (\partial_x h)^2 w^2 dx + 6k \left( \int hw^2 dx \right)^2$$

for  $\kappa = \frac{20}{3} \frac{\nu}{b}$ . Using  $\frac{c^2}{2\nu} \kappa = \kappa_*(a) \frac{\nu}{2}$ , we conclude that

$$\begin{aligned} \langle \nu \Delta u + b f'(v) u, u \rangle &\leq -(1 - \kappa_*(a)) \frac{\nu}{2} \int (\partial_x h)^2 w^2 dx \\ &\quad + \frac{b}{4} 6k \left( \int h w^2 dx \right)^2. \end{aligned}$$

This proves the first inequality.

For the proof of the second inequality observe that Lemma 4.2 again implies for any  $\gamma \in (0, 1)$  that

$$\begin{aligned} -(1 - \kappa_*(a)) \frac{\nu}{2} \int (\partial_x h)^2 w^2 dx &\leq -(1 - \gamma)(1 - \kappa_*(a)) \frac{\nu}{2} \int (\partial_x h)^2 w^2 dx \\ &\quad - \gamma \frac{\nu (1 - \kappa_*(a))}{2 \kappa} \|u\|_H^2 + \gamma \frac{\nu (1 - \kappa_*(a))}{2 \kappa} 6k \langle h, w^2 \rangle^2. \end{aligned}$$

Choosing  $\gamma = \frac{\kappa}{\nu + \kappa} = \frac{20}{20 + 3b}$  yields the inequality

$$\begin{aligned} \langle \nu \Delta u + b f'(v) u, u \rangle &\leq -\frac{3b(1 - \kappa_*(a))}{6b + 40} \left( \nu \int (\partial_x h)^2 w^2 dx + \|u\|_H^2 \right) \\ &\quad + \left( \frac{3b(1 - \kappa_*(a))}{6b + 40} + \frac{b}{4} \right) 6k \langle h, w^2 \rangle^2. \end{aligned}$$

Using the elementary estimate

$$\left( \frac{3b(1 - \kappa_*(a))}{6b + 40} + \frac{b}{4} \right) 6k \leq 3b \sqrt{\frac{b}{2\nu}}$$

we obtain the second estimate.  $\square$

The following Lemma has been used in the previous proof.

**Lemma 4.2.** *Let  $\kappa := \frac{20}{3} \frac{\nu}{b}$ . Then*

$$(29) \quad \int h^2 w^2 dx \leq \kappa \int (\partial_x h)^2 w^2 dx + Z^{-1} \left( \int h w^2 dx \right)^2$$

for all  $h \in C_c^1(\mathbb{R})$ . Here,  $Z = \int w^2 dx = \frac{1}{6k}$  is a normalizing constant.

*Proof.* We will first show that

$$(30) \quad \int (h - h(0))^2 w^2 dx \leq \kappa \int (\partial_x h)^2 w^2 dx.$$

To this end we will split up the estimate w.r.t.  $x \geq 0$  (resp.  $x \leq 0$ ) and show that

$$(31) \quad \int_0^\infty (h - h(0))^2 w^2 dx \leq \kappa \int_0^\infty (\partial_x h)^2 w^2 dx$$

and

$$(32) \quad \int_{-\infty}^0 (h - h(0))^2 w^2 dx \leq \kappa \int_{-\infty}^0 (\partial_x h)^2 w^2 dx.$$

Indeed note that for  $x \geq 0$ , using

$$\begin{aligned} (h(x) - h(0))^2 &= \left( \int_0^x \partial_x h(s) ds \right)^2 \leq \int_0^x e^{ks} ds \int_0^x e^{-ks} \partial_x h^2(s) ds \\ &= \frac{1}{k} (e^{kx} - 1) \int_0^x e^{-ks} \partial_x h^2(s) ds. \end{aligned}$$

Integrating the last inequality against  $w^2 dx$  we obtain that

$$\begin{aligned} \int_0^\infty (h - h(0))^2 w^2 dx &\leq \frac{1}{k} \int_0^\infty e^{-ks} \partial_x h^2(s) \int_s^\infty \frac{e^{-kx} - e^{-2kx}}{(1 + e^{-kx})^4} dx ds \\ &= \frac{1}{3k^2} \int_0^\infty e^{-ks} \partial_x h^2(s) \left( 1 - \frac{1 - 3e^{-ks}}{(1 + e^{-ks})^3} \right) ds \\ &\leq \frac{10}{3} \frac{1}{k^2} \int_0^\infty \partial_x h^2(s) w^2 ds \end{aligned}$$

which gives (31) because  $k^{-2} = \frac{2\nu}{b}$ .

For the proof of (32) note that  $w^2(-x) = w^2(x)$ , so that (31) implies (32). Clearly, combining (31) and (32) implies (30). For the final step of the proof of the Lemma we define the probability measure  $\mu(dx) := Z^{-1} w^2 dx$ . Then

$$\begin{aligned} \int h^2(x) \mu(dx) &= \text{Var}_\mu(h) + \left( \int h d\mu \right)^2 \\ &\leq Z^{-1} \int (h - h(0))^2 w^2 dx + \left( \int h d\mu \right)^2 \\ &\leq Z^{-1} \kappa \int (\partial_x h)^2 w^2 dx + \left( \int h d\mu \right)^2 \end{aligned}$$

which implies the assertion.  $\square$

**Lemma 4.3.** *Let  $u \in C_c^1(\mathbb{R})$  and write  $u = hw$ . Then*

$$\nu \int (\partial_x u)^2 dx + \int u^2 dx \leq (1 + b(\eta + 1)) \left( \nu \int (\partial_x h)^2 w^2 dx + \int u^2 dx \right).$$

*Proof.* Clearly,

$$\begin{aligned} \nu \int (\partial_x u)^2 dx &= \nu \int (\partial_x h)^2 w^2 dx + 2\nu \int h \partial_x h w \partial_x w dx + \nu \int (\partial_x w)^2 h^2 dx \\ &= \nu \int (\partial_x h)^2 w^2 dx - \nu \int h^2 \partial_{xx}^2 w w dx. \end{aligned}$$

Using the fact that

$$-\nu \partial_{xx}^2 w = b f'(v) w - c \nabla w \leq (b\eta + ck) w$$

and the obvious estimate  $ck = b(\frac{1}{2} - a) \leq b$ , we obtain the desired inequality.  $\square$

*Proof.* (of Theorem 2.3) First let  $u \in C_c^1(\mathbb{R})$  and note that under the assumption  $a > \frac{1}{2} \left(1 - \sqrt{\frac{3}{10}}\right)$ , hence  $\kappa_*(a) < 1$ , Proposition 4.1 implies the estimate

$$\begin{aligned} \langle \nu \Delta u + b f'(v) u, u \rangle &\leq -\frac{3b}{6b+40} (1 - \kappa_*(a)) \left( \nu \int (\partial_x h)^2 w^2 dx + \|u\|_H^2 \right) \\ &\quad + 3b \sqrt{\frac{b}{2\nu}} \langle u, \partial_x v \rangle^2. \end{aligned}$$

Combining the last estimate with the previous Lemma 4.3, we obtain that

$$\begin{aligned} \langle \nu \Delta u + b f'(v) u, u \rangle &\leq -\frac{3b}{6b+40} \frac{1}{1+b(\eta+1)} \left( \nu \int (\partial_x u)^2 dx + \|u\|_H^2 \right) \\ &\quad + 3b \sqrt{\frac{b}{2\nu}} \langle u, \partial_x v \rangle^2. \end{aligned}$$

This implies Theorem 2.3 with

$$\kappa_* = \frac{3b}{6b+40} \frac{1 - \kappa_*(a)}{1 + b(\eta+1)} \quad \text{and} \quad C_* = 3b \sqrt{\frac{b}{2\nu}}.$$

For general  $u \in V$  we deduce the desired estimate by standard approximation of  $u$  with  $C_c^1$ -functions in the  $V$ -norm.  $\square$

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